Linear Algebra and its Applications 433 (2010) 1388–1409

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

# Spectral characterization of some weighted rooted graphs with cliques☆

Oscar Rojo*[,1], Luis Medina

*Department of Mathematics, Universidad Católica del Norte, Antofagasta, Chile*

A R T I C L E  I N F O

*Article history:*
Received 13 April 2010
Accepted 18 May 2010
Available online 11 June 2010

Submitted by R.A. Brualdi

*AMS classification:*
5C50
15A48

*Keywords:*
Weighted graph
Laplacian matrix
Signless Laplacian matrix
Adjacency matrix
Generalized Bethe tree
Clique

A B S T R A C T

The level of a vertex in a rooted graph is one more than its distance from the root vertex. A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree. We characterize completely the eigenvalues of the Laplacian, signless Laplacian and adjacency matrices of a weighted rooted graph $\mathcal{G}$ obtained from a weighted generalized Bethe tree of $k$ levels and weighted cliques in which

(1) the edges connecting vertices at consecutive levels have the same weight,
(2) each set of children, in one or more levels, defines a weighted clique, and
(3) cliques at the same level are isomorphic.

These eigenvalues are the eigenvalues of symmetric tridiagonal matrices of order $j \times j$, $1 \leqslant j \leqslant k$. Moreover, we give results on the multiplicity of the eigenvalues, on the spectral radii and on the algebraic connectivity. Finally, we apply the results to the unweighted case and some particular graphs are studied.



☆ Work supported by Project Fondecyt 1100072, Chile.
* Corresponding author.
*E-mail address:* orojo@ucn.cl (O. Rojo).
[1] Part of this research was conducted while O. Rojo was a visitor at the Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile.

0024-3795/$ - see front matter 
doi:10.1016/j.laa.2010.05.015

## 1. Introduction

Let $\mathcal{G} = (V, E)$ be a simple undirected graph with vertex set $V$ and edge set $E$. We assume that each edge $e \in E$ has a positive weight $w(e)$. Let $V = \{1, 2, \ldots, n\}$. The Laplacian matrix $L(\mathcal{G}) = (l_{i,j})$, the signless Laplacian matrix $Q(\mathcal{G}) = (q_{i,j})$ and the adjacency matrix $A(\mathcal{G}) = (a_{i,j})$ of the graph $\mathcal{G}$, are the $n \times n$ matrices defined by

$$l_{i,j} = \begin{cases} -w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ -\sum_{k \neq i} l_{i,k} & \text{if } i = j, \end{cases}$$

$$q_{i,j} = \begin{cases} w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ \sum_{k \neq i} l_{i,k} & \text{if } i = j, \end{cases}$$

$$a_{i,j} = \begin{cases} w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ 0 & \text{if } i = j. \end{cases}$$

$L(\mathcal{G})$, $Q(\mathcal{G})$ and $A(\mathcal{G})$ are real symmetric matrices. From Geršgorin's Theorem, it follows that the eigenvalues of $L(\mathcal{G})$ and $Q(\mathcal{G})$ are nonnegative real numbers. Since the rows of $L(\mathcal{G})$ sum to 0, $(0, \mathbf{e})$ is an eigenpair for $L(\mathcal{G})$ where $\mathbf{e}$ is the all ones vector. Fiedler [8] proved that $\mathcal{G}$ is a connected graph if and only if the second smallest eigenvalue of $L(\mathcal{G})$ is positive. This eigenvalue, denoted by $a(\mathcal{G})$, is called the algebraic connectivity of $\mathcal{G}$. The signless Laplacian matrix has recently attracted the attention of some researchers. Recent papers on this matrix are [1–5].

If $w(e) = 1$ for all $e \in E$ then $\mathcal{G}$ is an unweighted graph. In [11], some of the many results known for the Laplacian matrix of an unweighted graph are given.

We recall that for a rooted graph the level of a vertex is one more than its distance from the root vertex. A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree.

Throughout this paper, let $\mathcal{B}$ be a weighted generalized Bethe tree of $k$ levels in which the vertices at the level $j$ have a degree equal to $d_{k-j+1} (1 \leqslant j \leqslant k)$ and the edges joining the vertices at the level $j$ with the vertices at the level $(j + 1)$ have a weight equal to $w_{k-j} (1 \leqslant j \leqslant k - 1)$.

In [12], we characterize completely the eigenvalues of the Laplacian and adjacency matrices of $\mathcal{B}$. In [7], the authors characterize the eigenvalues of the Laplacian and adjacency matrices of a graph obtained from $\mathcal{B}$ and equally weighted edges, in one or more levels, joining pairs of children.

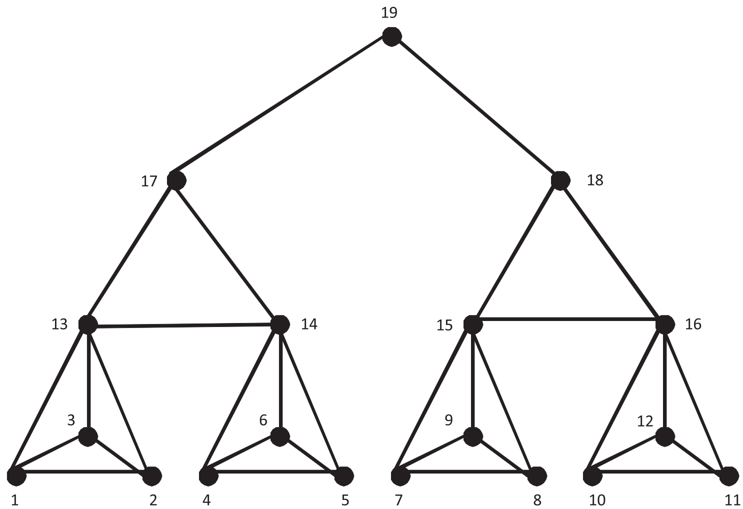
Let $\phi \neq \Delta \subseteq \{1, 2, \ldots, k - 1\}$. Our purpose is to search for the eigenvalues of the Laplacian, signless Laplacian and adjacency matrices of a weighted rooted graph $\mathcal{G}$ obtained from $\mathcal{B}$ and weighted cliques such that, for each $j \in \Delta$, the vertices of $\mathcal{G}$ at the level $k - j + 1$ connected with a vertex at the level $k - j$ define a clique in which the edges have a weight $u_{k-j+1}$. That is, for each $j \in \Delta$, the children of any vertex at the level $k - j$ define a clique in which the edges have weight $u_{k-j+1}$.

We assume $d_k > 1$ and $\mathcal{K}_p$ denotes the complete graph of $p$ vertices with $p \geqslant 2$.

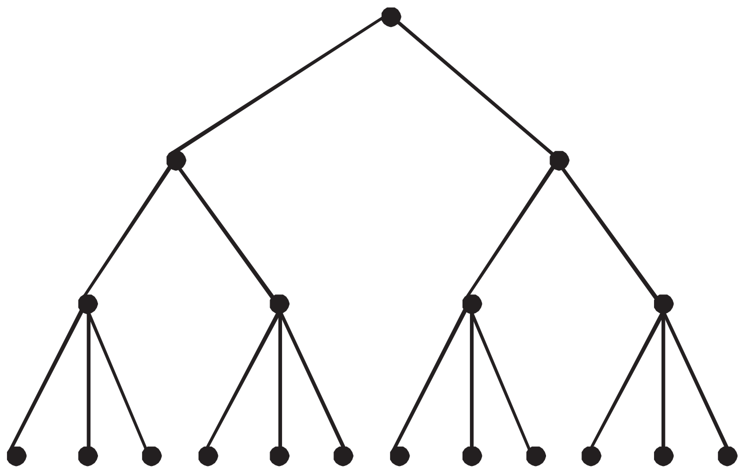
Observe that $\mathcal{B}$ and $\mathcal{G}$ have the same set of vertices and that the edges of $\mathcal{G}$ joining the vertices at the level $j$ with the vertices at the level $(j + 1)$ have a weight $w_{k-j} (1 \leqslant j \leqslant k - 1)$.

We label the vertices of $\mathcal{G}$ with $1, 2, \ldots, n$ from the level $k$ to the root vertex and, in each level, from the left to the right, and in each clique in a counterwise sense. Below is an example of our labelling.

**Example 1.** Let $\mathcal{G}$ be the graph




In this graph, $\mathcal{B}$ is the tree



There are $k = 4$ levels of vertices. We see that in $\mathcal{G}$ the vertices at the level 4 connected with a vertex at the level 3 define the clique $\mathcal{K}_3$ and the vertices at the levels 3 connected with a vertex at the level 2 define the clique $\mathcal{K}_2$.

We have already observed that $\mathcal{B}$ and $\mathcal{G}$ have the same set of vertices. We now observe that the degree of the vertices of $\mathcal{B}$ at the level $k - j + 1$ is $d_j$ and the degree of these vertices as vertices of $\mathcal{G}$ is $d_j + d_{j+1} - 2$ whenever $j \in \Delta$. For instance, at the level 3, the degree of the vertices of $\mathcal{B}$ in Example 1 have degree $d_2 = 4$ while the vertices of $\mathcal{G}$ have degree $d_2 + d_3 - 2 = 4 + 3 - 2 = 5$. Clearly, for $j \notin \Delta$ the vertices of $\mathcal{B}$ and $\mathcal{G}$ at the level $j$ have the same degree.

We introduce the following additional notation.

$|A|$ is the determinant of $A$.

For $j \notin \Delta$, $u_{k-j+1} = 0$.

For $j = 1, 2, \ldots, k$, let $n_{k-j+1}$ be the number of vertices at the level $j$ of $\mathcal{B}$. Observe that $n_k = 1$.

Let

$$\Omega = \left\{ j : 1 \leqslant j \leqslant k - 1,\ n_j > n_{j+1} \right\}.$$

For $j = 1, 2, \ldots, k-1$, let $m_j = \frac{n_j}{n_{j+1}}$. Clearly

$$n_j = (d_{j+1} - 1)\, n_{j+1},\ m_j - 1 = d_{j+1} - 2 \quad (1 \leqslant j \leqslant k-2),$$
$$n_{k-1} = d_k = m_{k-1}.$$

Let 0 and $I$ be the all zeros matrix and the identity of the appropriate order, respectively.

Let $I_m$ be the identity matrix of order $m \times m$ and let $\mathbf{e}_m$ be the $m$—dimensional column vector of ones.

Let

$$\begin{aligned}
\delta_1 &= w_1 + (d_2 - 2)\, u_1,\\
\delta_j &= (d_j - 1)\, w_{j-1} + w_j + (d_{j+1} - 2)\, u_j \quad (2 \leqslant j \leqslant k-2),\\
\delta_{k-1} &= (d_{k-1} - 1)\, w_{k-2} + w_{k-1} + (d_k - 1)\, u_{k-1},\\
\delta_k &= d_k w_{k-1}.
\end{aligned}$$

Observe that $\delta_j$ is the sum of the weights of the edges incident with the vertices at the level $k - j + 1$.

For the graph $\mathcal{G}$ in Example 1, we have $k = 4$, $\Delta = \{1, 2\}$,

$$\begin{aligned}
&d_1 = 1, \quad d_2 = 4, \quad d_3 = 3, \quad d_4 = 2,\\
&n_1 = 12, \quad n_2 = 4, \quad n_3 = 2, \quad n_4 = 1,\\
&\delta_1 = w_1 + 2u_1, \quad \delta_2 = 3w_1 + w_2 + u_2,\\
&\delta_3 = 2w_2 + w_3, \quad \delta_4 = 2w_3.
\end{aligned}$$

At this point, we recall that the Kronecker product [14] of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined to be the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}B)$. Some basic properties are

$$(A \otimes B)^T = A^T \otimes B^T$$

and

$$(A \otimes B)\,(C \otimes D) = (AC \otimes BD)$$

for matrices of appropriate sizes. In particular, if $A$ and $B$ are invertible matrices then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

The Laplacian, signless Laplacian and adjacency matrices of the graph $\mathcal{G}$ in Example 1 are, respectively:

$$L(\mathcal{G}) = \begin{bmatrix} I_4 \otimes (\delta_1 I_3 - u_1 A(\mathcal{K}_3)) & -w_1 I_4 \otimes \mathbf{e}_3 & 0 & 0\\ -w_1 I_4 \otimes \mathbf{e}_3^T & I_2 \otimes (\delta_2 I_2 - u_2 A(\mathcal{K}_2)) & -w_2 I_2 \otimes \mathbf{e}_2 & 0\\ 0 & -w_2 I_2 \otimes \mathbf{e}_2^T & \delta_3 I_2 & -w_3 \mathbf{e}_2\\ 0 & 0 & -w_3 \mathbf{e}_2^T & \delta_4 \end{bmatrix},$$

$$Q(\mathcal{G}) = \begin{bmatrix} I_4 \otimes (\delta_1 I_3 + u_1 A(\mathcal{K}_3)) & w_1 I_4 \otimes \mathbf{e}_3 & 0 & 0\\ w_1 I_4 \otimes \mathbf{e}_3^T & I_2 \otimes (\delta_2 I_2 + u_2 A(\mathcal{K}_2)) & w_2 I_2 \otimes \mathbf{e}_2 & 0\\ 0 & w_2 I_2 \otimes \mathbf{e}_2^T & \delta_3 I_2 & w_3 \mathbf{e}_2\\ 0 & 0 & w_3 \mathbf{e}_2^T & \delta_4 \end{bmatrix}$$

and

$$A(\mathcal{G}) = \begin{bmatrix} I_4 \otimes u_1 A(\mathcal{K}_3) & w_1 I_4 \otimes \mathbf{e}_3 & 0 & 0\\ w_1 I_4 \otimes \mathbf{e}_3^T & I_2 \otimes u_2 A(\mathcal{K}_2) & w_2 I_2 \otimes \mathbf{e}_2 & 0\\ 0 & w_2 I_2 \otimes \mathbf{e}_2^T & 0 & w_3 \mathbf{e}_2\\ 0 & 0 & w_3 \mathbf{e}_2^T & 0 \end{bmatrix}.$$

In general

$$L(\mathcal{G}) = \begin{bmatrix} I_{n_2} \otimes L_1 & -w_1 I_{n_2} \otimes \mathbf{e}_{m_1} & & & \\ -w_1 I_{n_2} \otimes \mathbf{e}_{m_1}^T & \ddots & \ddots & & \\ & \ddots & I_{n_{k-1}} \otimes L_{k-2} & -w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}} & \\ & & -w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}}^T & L_{k-1} & -w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & -w_{k-1}\mathbf{e}_{m_{k-1}}^T & \delta_k \end{bmatrix}$$

where

$$L_j = \delta_j I_{m_j} - u_j A\left(\mathcal{K}_{m_j}\right) \quad (1 \leqslant j \leqslant k-1),$$

$$Q(\mathcal{G}) = \begin{bmatrix} I_{n_2} \otimes Q_1 & w_1 I_{n_2} \otimes \mathbf{e}_{m_1} & & & \\ w_1 I_{n_2} \otimes \mathbf{e}_{m_1}^T & \ddots & \ddots & & \\ & \ddots & I_{n_{k-1}} \otimes Q_{k-2} & w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}} & \\ & & w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}}^T & Q_{k-1} & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & w_{k-1}\mathbf{e}_{m_{k-1}}^T & \delta_k \end{bmatrix},$$

where

$$Q_j = \delta_j I_{m_j} + u_j A\left(\mathcal{K}_{m_j}\right) \quad (1 \leqslant j \leqslant k-1),$$

and

$$A(\mathcal{G}) = \begin{bmatrix} I_{n_2} \otimes u_1 A\left(\mathcal{K}_{m_1}\right) & w_1 I_{n_2} \otimes \mathbf{e}_{m_1} & & & \\ w_1 I_{n_2} \otimes \mathbf{e}_{m_1}^T & \ddots & \ddots & & \\ & \ddots & I_{n_{k-1}} \otimes u_{k-2} A\left(\mathcal{K}_{m_{k-2}}\right) & w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}} & \\ & & w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}}^T & u_{k-1} A\left(\mathcal{K}_{m_{k-1}}\right) & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & w_{k-1}\mathbf{e}_{m_{k-1}}^T & 0 \end{bmatrix}.$$

Denote by

$$\lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \leqslant \lambda_n(M)$$

the eigenvalues of an $n \times n$ matrix $M$ with only real eigenvalues, $\sigma(M)$ the set of eigenvalues and $\rho(M)$ the spectral radius of a matrix $M$, respectively.

We finish this section collecting some well known facts in the following lemma.

**Lemma 1**

(a) *The eigenvalues of a Hermitian matrix do not decrease if a positive semidefinite matrix is added to it [10, Corollary 4.3.3].*
(b) *If $A$ is an $m \times m$ symmetric tridiagonal matrix with nonzero codiagonal entries then the eigenvalues of any $(m-1) \times (m-1)$ principal submatrix strictly interlace the eigenvalues of $A$ [9].*
(c) *A graph is bipartite if and only if it contains no odd cycle [6, Proposition 1.6.1] and that the smallest signless Laplacian eigenvalue of a connected graph is equal to 0 if and only if the graph is bipartite; in this case 0 is a simple eigenvalue [2, Proposition 2.1].*

(d) *If $M$ is an irreducible nonnegative matrix then $\rho(M)$ strictly increases when any entry of $M$ strictly increases [13, Theorem 2.1].*

## 2. The spectrum of $L(\mathcal{G})$

**Lemma 2.** *Let*

$$B = \beta I_m + uA\left(\mathcal{K}_m\right).$$

*Then*

$$|B| = (\beta - u)^{m-1}\left(\beta + (m-1)\,u\right). \tag{1}$$

*and if $\beta - u \neq 0$ and $\beta + (m-1)u \neq 0$,*

$$\mathbf{e}_m^T B^{-1}\mathbf{e}_m = \frac{m}{\beta + (m-1)\,u}.$$

**Proof.** The eigenvalues of $B$ are $\beta - u$ with multiplicity $m-1$ and $\beta + u(m-1)$. Thus (1) follows easily. If $\beta - u \neq 0$ and $\beta + (m-1)u \neq 0$ then $B$ is invertible. One can verify that

$$B^{-1} = xI_m + yA\left(\mathcal{K}_m\right)$$

where

$$x = \frac{\beta + (m-2)\,u}{(\beta - u)\left(\beta + (m-1)\,u\right)}, \quad y = \frac{-u}{(\beta - u)\left(\beta + (m-1)\,u\right)}.$$

Hence

$$\begin{aligned}\mathbf{e}_m^T B^{-1}\mathbf{e}_m &= mx + (m-1)\,my \\ &= \frac{m}{\beta + (m-1)\,u}. \quad \square\end{aligned}$$

**Lemma 3.** *Let*

$$A_j = \alpha_j I_{m_j} + u_j A\left(\mathcal{K}_{m_j}\right) \quad (1 \leqslant j \leqslant k-1).$$

*Let $M$ be the block tridiagonal matrix*

$$M = \begin{bmatrix} I_{n_2}\otimes A_1 & w_1 I_{n_2}\otimes \mathbf{e}_{m_1} & & & & \\ w_1 I_{n_2}\otimes \mathbf{e}_{m_1}^T & I_{n_3}\otimes A_2 & w_2 I_{n_3}\otimes \mathbf{e}_{m_2} & & & \\ & w_2 I_{n_3}\otimes \mathbf{e}_{m_2}^T & \ddots & \ddots & & \\ & & \ddots & I_{n_{k-1}}\otimes A_{k-2} & w_{k-2} I_{n_{k-1}}\otimes \mathbf{e}_{m_{k-2}} & \\ & & & w_{k-2} I_{n_{k-1}}\otimes \mathbf{e}_{m_{k-2}}^T & A_{k-1} & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & & w_{k-1}\mathbf{e}_{m_{k-1}}^T & \alpha_k \end{bmatrix}.$$

*Let*

$$\beta_1 = \alpha_1$$

*and, for $j = 1, 2, \ldots, k-1$, if $\beta_j - u_j \neq 0$ and $\beta_j + u_j(m_j - 1) \neq 0$, let*

$$\beta_{j+1} = \alpha_{j+1} - \frac{m_j w_j^2}{\beta_j + u_j\left(m_j - 1\right)}.$$

*Then*

$$|M| = \beta_k \prod_{j=1}^{k-1}\left(\beta_j - u_j\right)^{n_j - n_{j+1}}\left(\beta_j + \left(m_j - 1\right)u_j\right)^{n_{j+1}}. \tag{2}$$

**Proof.** In order to prove (2), we reduce $M$ to a block upper triangular matrix. We have $A_1 = \alpha_1 I_{m_1} + u_1 A(\mathcal{K}_{m_1}) = \beta_1 I_{m_1} + u_1 A(\mathcal{K}_{m_1}) \equiv B_1$. Since $\beta_1 - u_1 \neq 0$ and $\beta_1 + u_1(m_1 - 1) \neq 0$, from Lemma 2, the matrix $B_1$ is invertible. Multiplying the first row of blocks by $w_1 I_{n_2} \otimes \mathbf{e}_{m_1}^T B_1^{-1}$ and subtracting the products from the second row of blocks, we obtain

$$M_2 = \begin{bmatrix} I_{n_2} \otimes B_1 & w_1 I_{n_2} \otimes \mathbf{e}_{m_1} & & & & \\ & I_{n_3} \otimes A_2 - w_1^2 I_{n_2} \otimes \mathbf{e}_{m_1}^T B_1^{-1} \mathbf{e}_{m_1} & w_2 I_{n_3} \otimes \mathbf{e}_{m_2} & & & \\ & w_2 I_{n_3} \otimes \mathbf{e}_{m_2}^T & \ddots & \ddots & & \\ & & \ddots & & \ddots & \\ & & & \ddots & A_{k-1} & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & & w_{k-1}\mathbf{e}_{m_{k-1}}^T & \alpha_k \end{bmatrix}.$$

From Lemma 2, $\mathbf{e}_{m_1}^T B_1^{-1} \mathbf{e}_{m_1} = \frac{m_1}{\beta_1 + u_1(m_1-1)}$. Then

$$\begin{aligned} & I_{n_3} \otimes A_2 - w_1^2 I_{n_2} \otimes \mathbf{e}_{m_1}^T B_1^{-1} \mathbf{e}_{m_1} \\ &= I_{n_3} \otimes A_2 - w_1^2 I_{n_2} \otimes \frac{m_1}{\beta_1 + u_1 (m_1 - 1)} \\ &= I_{n_3} \otimes A_2 - \frac{m_1 w_1^2}{\beta_1 + u_1 (m_1 - 1)} I_{n_2} \\ &= \begin{bmatrix} A_2 - \frac{m_1 w_1^2}{\beta_1 + u_1 (m_1-1)} I_{m_2} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & A_2 - \frac{m_1 w_1^2}{\beta_1 + u_1(m_1-1)} I_{m_2} \end{bmatrix} \\ &= I_{n_3} \otimes \left( A_2 - \frac{m_1 w_1^2}{\beta_1 + u_1 (m_1 - 1)} I_{m_2} \right) \\ &= I_{n_3} \otimes \left( \left( \alpha_2 - \frac{m_1 w_1^2}{\beta_1 + u_1 (m_1 - 1)} \right) I_{m_2} + u_2 A \left( \mathcal{K}_{m_2} \right) \right) \\ &= I_{n_3} \otimes \left( \beta_2 I_{m_2} + u_2 A \left( \mathcal{K}_{m_2} \right) \right). \end{aligned}$$

Let $B_2 = \beta_2 I_{m_2} + u_2 A(\mathcal{K}_{m_2})$. Thus

$$\begin{aligned} |M| &= |M_2| \\ &= \begin{vmatrix} I_{n_2} \otimes B_1 & w_1 I_{n_2} \otimes \mathbf{e}_{m_1} & & & & & \\ 0 & I_{n_3} \otimes B_2 & w_2 I_{n_3} \otimes \mathbf{e}_{m_2} & & & & \\ & w_2 I_{n_3} \otimes \mathbf{e}_{m_2}^T & \ddots & & \ddots & & \\ & & \ddots & I_{n_{k-1}} \otimes A_{k-2} & w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}} & \\ & & & w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}}^T & A_{k-1} & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & & w_{k-1}\mathbf{e}_{m_{k-1}}^T & \alpha_k \end{vmatrix}. \end{aligned}$$

Since $\beta_2 - u_2 \neq 0$ and $\beta_2 + u_2(m_2 - 1) \neq 0$, from Lemma 2, the matrix $B_2$ is invertible and we can continue with our procedure. Just before the last step, we obtain

$$|M| = |M_{k-1}| = \begin{vmatrix} I_{n_2}\otimes B_1 & w_1 I_{n_2}\otimes \mathbf{e}_{m_1} & & & & \\ 0 & I_{n_3}\otimes B_2 & w_2 I_{n_3}\otimes \mathbf{e}_{m_2} & & & \\ & 0 & \ddots & \ddots & & \\ & & \ddots & I_{n_{k-1}}\otimes B_{k-2} & w_{k-2} I_{n_{k-1}}\otimes \mathbf{e}_{m_{k-2}} & \\ & & & 0 & B_{k-1} & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & & w_{k-1}\mathbf{e}^T_{m_{k-1}} & \alpha_k \end{vmatrix}$$

where $B_j = \beta_j I_{m_j} + u_{m_j} A(\mathcal{K}_{m_j})$ for $j = 1, 2, \ldots, k-1$. Finally, the procedure gives

$$|M| = \begin{vmatrix} I_{n_2}\otimes B_1 & w_1 I_{n_2}\otimes \mathbf{e}_{m_1} & & & & \\ 0 & I_{n_3}\otimes B_2 & w_2 I_{n_3}\otimes \mathbf{e}_{m_2} & & & \\ & 0 & \ddots & \ddots & & \\ & & \ddots & I_{n_{k-1}}\otimes B_{k-2} & w_{k-2} I_{n_{k-1}}\otimes \mathbf{e}_{m_{k-2}} & \\ & & & 0 & B_{k-1} & w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & & 0 & \beta_k \end{vmatrix}$$

with $\beta_k = \alpha_k - \frac{m_{k-1}w_{k-1}^2}{\beta_{k-1}+u_{k-1}(m_{k-1}-1)}$. Hence

$$|M| = \beta_k \prod_{j=1}^{k-1} |B_j|^{n_{j+1}} .$$

From Lemma 2

$$|B_j| = (\beta_j - u_j)^{m_j-1} (\beta_j + (m_j-1) u_j) .$$

Therefore

$$|M| = \beta_k \prod_{j=1}^{k-1} (\beta_j - u_j)^{n_j-n_{j+1}} (\beta_j + (m_j-1) u_j)^{n_{j+1}} .$$

This completes the proof. □

**Definition 1.** Let

$$C_0(\lambda) = 1, \quad C_1(\lambda) = \lambda - w_1,$$
$$C_j(\lambda) = (\lambda - (d_j-1) w_{j-1} - w_j) C_{j-1}(\lambda) - m_{j-1}w_{j-1}^2 C_{j-2}(\lambda)$$

for $j = 2, 3, \ldots, k-1$, and

$$C_k(\lambda) = (\lambda - d_k w_{k-1}) C_{k-1}(\lambda) - m_{k-1}w_{k-1}^2 C_{k-2}(\lambda) .$$

**Theorem 1.** *The characteristic polynomial of $L(\mathcal{G})$ is*

$$|\lambda I - L(\mathcal{G})| = C_k(\lambda) \prod_{j\in\Omega} (C_j(\lambda) - m_j u_j C_{j-1}(\lambda))^{n_j-n_{j+1}} . \quad (3)$$

*Then*

$$|\lambda I - L(\mathcal{G})| = C_k(\lambda) \prod_{j\in\Omega-\Delta} C_j^{n_j-n_{j+1}}(\lambda) \prod_{j\in\Delta} (C_j(\lambda) - m_j u_j C_{j-1}(\lambda))^{n_j-n_{j+1}} . \quad (4)$$

**Proof.** For brevity, we write sometimes $C_j$ instead of $C_j(\lambda)$. Let $\lambda \in \mathbb{R}$ such that $C_j(\lambda) \neq 0$ for $j = 1, 2, \ldots, k-1$. Applying Lemma 3 to $\lambda I - L(\mathcal{G})$, we have

$$\begin{aligned}\beta_1 &= \alpha_1 = \lambda - \delta_1 = \lambda - w_1 - (d_2 - 2)\, u_1 \\ &= \lambda - w_1 - (m_1 - 1)\, u_1 = C_1 - (m_1 - 1)\, u_1.\end{aligned}$$

Then

$$\begin{aligned}\beta_1 - u_1 &= C_1 - m_1 u_1 = \frac{C_1}{C_0} - m_1 u_1, \\ \beta_1 + (m_1 - 1)\, u_1 &= \frac{C_1}{C_0} \neq 0\end{aligned}$$

and thus

$$\begin{aligned}\beta_2 &= \alpha_2 - \frac{m_1 w_1^2}{\beta_1 + (m_1 - 1)\, u_1} \\ &= \lambda - \delta_2 - \frac{m_1 w_1^2}{\beta_1 + (m_1 - 1)\, u_1} \\ &= \lambda - (d_2 - 1)\, w_1 - w_2 - (d_3 - 2)\, u_2 - \frac{m_1 w_1^2 C_0}{C_1} \\ &= \frac{(\lambda - (d_2 - 1)\, w_1 - w_2)\, C_1 - m_1 w_1^2 C_0}{C_1} - (m_2 - 1)\, u_2 \\ &= \frac{C_2}{C_1} - (m_2 - 1)\, u_2.\end{aligned}$$

Then

$$\begin{aligned}\beta_2 - u_2 &= \frac{C_2}{C_1} - m_2 u_2 \\ \beta_2 + (m_2 - 1)\, u_2 &= \frac{C_2}{C_1} \neq 0.\end{aligned}$$

Similarly, for $j = 3, 4, \ldots, k-1$

$$\begin{aligned}\beta_j - u_j &= \frac{C_j}{C_{j-1}} - m_j u_j \\ \beta_j + (m_j - 1)\, u_j &= \frac{C_j}{C_{j-1}} \neq 0.\end{aligned}$$

In addition

$$\begin{aligned}\beta_k &= \alpha_k - \frac{m_{k-1} w_{k-1}^2}{\beta_{k-1} + (m_{k-1} - 1)\, u_{k-1}} \\ &= \lambda - d_k w_{k-1} - \frac{m_{k-1} w_{k-1}^2 C_{k-2}}{C_{k-1}} \\ &= \frac{(\lambda - d_k w_{k-1})\, C_{k-1} - m_{k-1} w_{k-1}^2 C_{k-2}}{C_{k-1}} \\ &= \frac{C_k}{C_{k-1}}.\end{aligned}$$

Replacing these results in (2), we obtain

$$\begin{aligned}
&|\lambda I - L(\mathcal{G})| \\
&= \frac{C_k}{C_{k-1}} \prod_{j=1}^{k-1} \left( \frac{C_j}{C_{j-1}} - m_j u_j \right)^{n_j - n_{j+1}} \left( \frac{C_j}{C_{j-1}} \right)^{n_{j+1}} \\
&= \frac{C_k}{C_{k-1}} \prod_{j=1}^{k-1} \left( \frac{C_j - m_j u_j C_{j-1}}{C_{j-1}} \right)^{n_j - n_{j+1}} \left( \frac{C_j}{C_{j-1}} \right)^{n_{j+1}} \\
&= \frac{C_k}{C_{k-1}} \prod_{j=1}^{k-1} \left( C_j - m_j u_j C_{j-1} \right)^{n_j - n_{j+1}} \prod_{j=1}^{k-1} \frac{C_j^{n_{j+1}}}{C_{j-1}^{n_j}} \\
&= C_k \prod_{j=1}^{k-1} \left( C_j - m_j u_j C_{j-1} \right)^{n_j - n_{j+1}} \\
&= C_k \prod_{j \in \Omega} \left( C_j - m_j u_j C_{j-1} \right)^{n_j - n_{j+1}}.
\end{aligned}$$

Thus (3) is proved for all $\lambda \in \mathbb{R}$ such that $C_j(\lambda) \neq 0$ for $j = 1, 2, \ldots, k-1$. Now, we consider $\lambda_0 \in \mathbb{R}$ such that $C_s(\lambda_0) = 0$ for some $1 \leqslant s \leqslant k-1$. Since the zeros of any nonzero polynomial are isolated, there exists a neighborhood $N(\lambda_0)$ of $\lambda_0$ such that $C_j(\lambda) \neq 0$ for all $\lambda \in N(\lambda_0) - \{\lambda_0\}$ and for all $j = 1, 2, \ldots, k-1$. Hence

$$|\lambda I - L(\mathcal{G})| = C_k(\lambda) \prod_{j \in \Omega} \left( C_j(\lambda) - m_j u_j C_{j-1}(\lambda) \right)^{n_j - n_{j+1}}$$

for all $\lambda \in N(\lambda_0) - \{\lambda_0\}$. By continuity, taking the limit as $\lambda$ tends to $\lambda_0$, we obtain

$$|\lambda_0 I - L(\mathcal{G})| = C_k(\lambda_0) \prod_{j \in \Omega} \left( C_j(\lambda_0) - m_j u_j C_{j-1}(\lambda_0) \right)^{n_j - n_{j+1}}.$$

Therefore (3) holds for all $\lambda \in \mathbb{R}$. We observe that if $j \in \Delta$ then $n_j > n_{j+1}$, that is, $\Delta \subseteq \Omega$. Thus $\Omega \cap \Delta = \Delta$ and since $u_j = 0$ for all $j \notin \Delta$, (4) follows from (3). $\square$

**Definition 2.** For $j = 1, 2, 3, \ldots, k-1$, let $U_j$ be the $j \times j$ leading principal submatrix of the $k \times k$ matrix

$$U_k = \begin{bmatrix}
w_1 & \sqrt{m_1} w_1 & & & \\
\sqrt{m_1} w_1 & (d_2 - 1) w_1 + w_2 & \ddots & & \\
& \ddots & \ddots & \sqrt{m_{k-2}} w_{k-2} & \\
& & \sqrt{m_{k-2}} w_{k-2} & (d_{k-1} - 1) w_{k-2} + w_{k-1} & \sqrt{m_{k-1}} w_{k-1} \\
& & & \sqrt{m_{k-1}} w_{k-1} & d_k w_{k-1}
\end{bmatrix}.$$

One can easily prove that $|U_j| = w_1 w_2 \cdots w_j$ for $j = 1, 2, \ldots, k-1$ and $|U_k| = 0$. Then $U_j$ is a positive definite matrix for $j = 1, 2, \ldots, k-1$.

The next Lemma gives the relationship between the polynomials $C_j$ and the matrices $U_j$.

**Lemma 4.** *For* $j = 1, 2, \ldots, k$,

$$|\lambda I - U_j| = C_j. \tag{5}$$

**Proof.** Clearly

$$|\lambda I - U_1| = \lambda - w_1 = C_1.$$

and

$$\begin{aligned}|\lambda I-U_2| &= \begin{vmatrix}\lambda-w_1 & -\sqrt{m_1}w_1\\ -\sqrt{m_1}w_1 & \lambda-(d_2-1)\,w_1-w_2\end{vmatrix}\\ &=(\lambda-w_1)\,(\lambda-(d_2-1)\,w_1-w_2)-m_1w_1^2\\ &=(\lambda-(d_2-1)\,w_1-w_2)\,C_1-m_1w_1^2C_0=C_2.\end{aligned}$$

Let $3\leqslant j\leqslant k-1$. Suppose that (5) holds for all $j<3$. We have

$$\begin{aligned}&\left|\lambda I-U_j\right|\\ &=\begin{vmatrix}\lambda-w_1 & -\sqrt{m_1}w_1 & & & \\ -\sqrt{m_1}w_1 & \ddots & \ddots & & \\ & \ddots & \ddots & -\sqrt{m_{j-2}}w_{j-2} & \\ & & -\sqrt{m_{j-2}}w_{j-2} & \ddots & -\sqrt{m_{j-1}}w_{j-1}\\ & & & -\sqrt{m_{j-1}}w_{j-1} & \lambda-(d_j-1)\,w_{j-1}-w_j\end{vmatrix}.\end{aligned}$$

We expand about the last row. Clearly the cofactor for $\lambda-(d_j-1)w_{j-1}-w_j$ is $|\lambda I-U_{j-1}|$. The cofactor for the entry $-\sqrt{m_{j-1}}w_{j-1}$ is

$$\begin{aligned}&-\begin{vmatrix}\lambda-w_1 & -\sqrt{m_1}w_1 & & \\ -\sqrt{m_1}w_1 & \lambda-(d_2-1)\,w_1-w_2 & \ddots & \\ & \ddots & \ddots & 0\\ & & -\sqrt{m_{j-2}}w_{j-2} & -\sqrt{m_{j-1}}w_{j-1}\end{vmatrix}\\ &=-\sqrt{m_{j-1}}w_{j-1}\left|\lambda I-U_{j-2}\right|.\end{aligned}$$

Therefore

$$\left|\lambda I-U_j\right|=\left(\lambda-\left(d_j-1\right)w_{j-1}-w_j\right)\left|\lambda I-U_{j-1}\right|-m_{j-1}w_{j-1}^2\left|\lambda I-U_{j-2}\right|.$$

We use the induction hypothesis to obtain

$$\left|\lambda I-U_j\right|=\left(\lambda-\left(d_j-1\right)w_{j-1}-w_j\right)C_{j-1}-m_{j-1}w_{j-1}^2C_{j-2}=C_j.$$

Finally, expanding about the last row of $|\lambda I-U_k|$ and using the facts $|\lambda I-U_{k-2}|=C_{k-2}$ and $|\lambda I-U_{k-1}|=C_{k-1}$, we get $|\lambda I-U_k|=C_k$. Thus the proof is complete. $\square$

We now study the polynomials

$$D_j=C_j-m_ju_jC_{j-1}$$

which appear in (4). We have

$$\begin{aligned}D_1&=\lambda-w_1-m_1u_1,\\ D_j&=\left(\lambda-\left(d_j-1\right)w_{j-1}-w_j-m_ju_j\right)C_{j-1}-m_{j-1}w_{j-1}^2C_{j-2}\end{aligned}$$

for $j=2,3,\ldots,k-1$.

**Definition 3.** Let

$$\begin{aligned}V_1&=w_1+(d_2-1)\,u_1\\ V_j&=\begin{bmatrix}U_{j-1} & \mathbf{v}_{j-1}\\ \mathbf{v}_{j-1}^T & \left(d_j-1\right)w_{j-1}+w_j+m_ju_j\end{bmatrix}\\ &=U_j+\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0}^T & m_ju_j\end{bmatrix}\quad\left(2\leqslant j\leqslant k-1\right),\end{aligned}$$

where $\mathbf{v}_{j-1}$ is the $(j-1)$-dimensional vector $\mathbf{v}_{j-1}=\begin{bmatrix}0 & \cdots & 0 & \sqrt{m_{j-1}}w_{j-1}\end{bmatrix}^T$.

From Lemma 1, part $(a)$, we have

$$\lambda_i\left(U_j\right) \leqslant \lambda_i\left(V_j\right) \tag{6}$$

for $j = 1, 2, \ldots, k-1$ and $i = 1, 2, \ldots, j$. Observe that $V_j = U_j$ for $j \notin \Delta$.

**Lemma 5.** *For* $j = 1, 2, \ldots, k-1$

$$\left|\lambda I - V_j\right| = D_j.$$

**Proof.** Similar to the proof of Lemma 4. □

An immediate consequence of Theorem 1, Lemmas 4 and 5 is the next theorem which gives a complete characterization of the eigenvalues of $L(\mathcal{G})$ together with some results on their multiplicities.

**Theorem 2**

(a) $\sigma\left(L\left(\mathcal{G}\right)\right) = \left(\cup_{j \in \Omega - \Delta} \sigma\left(U_j\right)\right) \cup \sigma\left(U_k\right) \cup \left(\cup_{j \in \Delta} V_j\right).$

(b) *For each* $j \in \Omega - \Delta$, *the multiplicity of the eigenvalues of* $U_j$ *and, for each* $j \in \Delta$, *the multiplicity of the eigenvalues of* $V_j$, *as eigenvalues of* $L(\mathcal{G})$, *is at least* $(n_j - n_{j+1})$. *The multiplicity of the eigenvalues of* $U_k$ *is at least* 1.

**Corollary 1**

(a) 0 *is an eigenvalue of* $U_k$.

(b) $$\rho\left(L\left(\mathcal{G}\right)\right) = \max\left\{\rho\left(U_k\right), \max_{j \in \Delta}\left\{\rho\left(V_j\right)\right\}\right\}. \tag{7}$$

(c) *If* $k-1 \in \Omega - \Delta$ *then the algebraic connectivity of* $\mathcal{G}$, $a(\mathcal{G})$, *is the smallest eigenvalue of* $U_{k-1}$.

**Proof**

(a) From Lemma 1, part $(b)$, $\lambda_1(U_k) < \lambda_1(U_j)$ for $j = 1, 2, \ldots, k-1$. From (6), $\lambda_1(U_j) \leqslant \lambda_1(V_j)$. Then $\lambda_1(U_k)$ is the smallest Laplacian eigenvalue. Therefore $\lambda_1(U_k) = 0$.

(b) From Lemma 1, part $(b)$, $\rho(U_j) < \rho(U_k)$ for $j = 1, 2, \ldots, k-1$. Thus (7) follows.

(c) Suppose $k-1 \in \Omega - \Delta$. Then $\lambda_1(U_{k-1})$ is a Laplacian eigenvalue. From Lemma 1, part $(b)$, and (6),

$$\lambda_1\left(U_{k-1}\right) < \lambda_1\left(U_j\right) \leqslant \lambda_1\left(V_j\right)$$

for $j = 1, 2, \ldots, k-2$, and

$$0 = \lambda_1\left(U_k\right) < \lambda_1\left(U_{k-1}\right) < \lambda_2\left(U_k\right).$$

Therefore $\lambda_1(U_{k-1}) = a(\mathcal{G})$ if $k-1 \in \Omega - \Delta$. □

**Example 2.** For the graph $\mathcal{G}$ in Example 1, $k = 4$, $\Delta = \{1, 2\}$, $d_1 = 1$, $d_2 = 4$, $d_3 = 3$, $d_4 = 2$, $n_1 = 12$, $n_2 = 4$, $n_3 = 2$, $n_4 = 1$ and $\Omega = \{1, 2, 3\}$. Then

$$\sigma\left(L\left(\mathcal{G}\right)\right) = \sigma\left(U_3\right) \cup \sigma\left(U_4\right) \cup \sigma\left(V_1\right) \cup \sigma\left(V_2\right)$$

where

$$U_3 = \begin{bmatrix} w_1 & \sqrt{3}w_1 & \\ \sqrt{3}w_1 & 3w_1 + w_2 & \sqrt{2}w_2 \\ & \sqrt{2}w_2 & 2w_2 + w_3 \end{bmatrix},$$

$$U_4 = \begin{bmatrix} w_1 & \sqrt{3}w_1 & & \\ \sqrt{3}w_1 & 3w_1 + w_2 & \sqrt{2}w_2 & \\ & \sqrt{2}w_2 & 2w_2 + w_3 & \sqrt{2}w_3 \\ & & \sqrt{2}w_3 & 2w_3 \end{bmatrix},$$

$$V_1 = \left[w_1 + 3u_1\right], \; V_2 = \begin{bmatrix} w_1 & \sqrt{3}w_1 \\ \sqrt{3}w_1 & 3w_1 + w_2 + 2u_2 \end{bmatrix}.$$

For $w_1 = 2,\ w_2 = 2.5,\ w_3 = 3,\ u_1 = 1.5$ and $u_2 = 2$, the eigenvalues of $L(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $U_3:$ | 0.2063 | 5.8395 | 12.4542 | | $n_3 - n_4 = 1$ |
| $U_4:$ | 0 | 2.4811 | 8.4917 | 13.5272 | 1 |
| $V_1:$ | 6.5 | | | | $n_1 - n_2 = 8$ |
| $V_2:$ | 0.9601 | 13.5399 | | | $n_2 - n_3 = 2$ |

## 3. The spectra of $Q(\mathcal{G})$ and $A(\mathcal{G})$

**Definition 4.** Let

$$E_0(\lambda) = 1, \quad E_1(\lambda) = \lambda - w_1 - 2(m_1 - 1)u_1,$$
$$E_j(\lambda) = (\lambda - (d_j - 1)w_{j-1} - w_j - 2(m_j - 1)u_j)E_{j-1}(\lambda) - m_{j-1}w_{j-1}^2 E_{j-2}(\lambda)$$

for $j = 2, 3, \ldots, k-1$, and

$$E_k = (\lambda - d_k w_{k-1})E_{k-1} - m_{k-1}w_{k-1}^2 E_{k-2}.$$

**Lemma 6.** *Let*

$$B = \beta I_m - uA(\mathcal{K}_m).$$

*Then*

$$|B| = (\beta + u)^{m-1}(\beta - (m-1)u).$$

*and if $\beta + u \neq 0$ and $\beta - (m-1)u \neq 0$,*

$$\mathbf{e}_m^T B^{-1}\mathbf{e}_m = \frac{m}{\beta - (m-1)u}.$$

**Proof.** Similar to the proof of Lemma 2. □

**Lemma 7.** *Let*

$$A_j = \alpha_j I_{m_j} - u_j A\left(\mathcal{K}_{m_j}\right) \quad (1 \leqslant j \leqslant k-1).$$

*Let M be the block tridiagonal matrix*

$$M = \begin{bmatrix} I_{n_2} \otimes A_1 & -w_1 I_{n_2} \otimes \mathbf{e}_{m_1} & & & & \\ -w_1 I_{n_2} \otimes \mathbf{e}_{m_1}^T & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & I_{n_{k-1}} \otimes A_{k-2} & -w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}} & \\ & & & -w_{k-2} I_{n_{k-1}} \otimes \mathbf{e}_{m_{k-2}}^T & A_{k-1} & -w_{k-1}\mathbf{e}_{m_{k-1}} \\ & & & & -w_{k-1}\mathbf{e}_{m_{k-1}}^T & \alpha_k \end{bmatrix}.$$

*Let*

$$\beta_1 = \alpha_1$$

*and, for $j = 1, 2, \ldots, k-1$, $\beta_j + u_j \neq 0$ and $\beta_j - u_j(m_j - 1) \neq 0$, let*

$$\beta_{j+1} = \alpha_{j+1} - \frac{m_j w_j^2}{\beta_j - u_j(m_j - 1)}.$$

*Then*

$$|M| = \beta_k \prod_{j=1}^{k-1} \left(\beta_j + u_j\right)^{n_j - n_{j+1}} \left(\beta_j - \left(m_j - 1\right) u_j\right)^{n_{j+1}} . \tag{8}$$

**Proof.** Similar to the proof of Lemma 3. $\square$

**Theorem 3.** *The characteristic polynomial of* $Q(\mathcal{G})$ *is*

$$|\lambda I - Q(\mathcal{G})| = E_k(\lambda) \prod_{j\in\Omega} \left(E_j(\lambda) + m_j u_j E_{j-1}(\lambda)\right)^{n_j - n_{j+1}} .$$

*Then*

$$|\lambda I - Q(\mathcal{G})| = E_k(\lambda) \prod_{j\in\Omega-\Delta} E_j^{n_j - n_{j+1}}(\lambda) \prod_{j\in\Delta} \left(E_j(\lambda) + m_j u_j E_{j-1}(\lambda)\right)^{n_j - n_{j+1}} . \tag{9}$$

**Proof.** Similar to the proof of Theorem 1. $\square$

**Definition 5.** For $j = 1, 2, 3, \ldots, k-1$, let $W_j$ be the $j \times j$ leading principal submatrix of the $k \times k$ matrix

$$W_k = \begin{bmatrix} w_1 + 2(m_1 - 1)u_1 & \sqrt{m_1} w_1 & & & \\ \sqrt{m_1} w_1 & W_k(2,2) & \ddots & & \\ & \ddots & \ddots & \sqrt{m_{k-2}} w_{k-2} & \\ & & \sqrt{m_{k-2}} w_{k-2} & W_k(k-1,k-1) & \sqrt{m_{k-1}} w_{k-1} \\ & & & \sqrt{m_{k-1}} w_{k-1} & d_k w_{k-1} \end{bmatrix},$$

where

$$W_k(j,j) = \left(d_j - 1\right) w_{j-1} + w_j + 2\left(m_j - 1\right) u_j \quad (2 \leqslant j \leqslant k-1) .$$

**Lemma 8.** *For* $j = 1, 2, \ldots, k$,

$$\left|\lambda I - W_j\right| = E_j .$$

**Proof.** Similar to the proof of Lemma 4. $\square$

We now study the polynomials

$$F_j = E_j + m_j u_j E_{j-1}$$

which appear in (9). We have

$$\begin{aligned} F_1 &= \lambda - w_1 - 2(d_2 - 2)u_1 + (d_2 - 1)u_1 \\ &= \lambda - w_1 - (d_2 - 3)u_1 . \end{aligned}$$

$$\begin{aligned} F_j &= \left(\lambda - \left(d_j - 1\right) w_{j-1} - w_j - 2\left(m_j - 1\right) u_j\right) E_{j-1} - m_{j-1} w_{j-1}^2 E_{j-2} + m_j u_j E_{j-1} \\ &= \left(\lambda - \left(d_j - 1\right) w_{j-1} - w_j - \left(m_j - 2\right) u_j\right) E_{j-1} - m_{j-1} w_{j-1}^2 E_{j-2} \end{aligned}$$

for $j = 2, 3, \ldots, k-1$.

**Definition 6.** Let

$$X_1 = w_1 + (d_2 - 3)\, u_1$$

$$X_j = \begin{bmatrix} W_{j-1} & \mathbf{v}_{j-1} \\ \mathbf{v}_{j-1}^T & (d_j - 1)\, w_{j-1} + w_j + (m_j - 2)\, u_j \end{bmatrix} = W_j - \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & m_j u_j \end{bmatrix} \ (2 \leqslant j \leqslant k-1)$$

where $\mathbf{v}_{j-1}$ is the $(j-1)$-dimensional vector $\mathbf{v}_{j-1} = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{m_{j-1}} w_{j-1} \end{bmatrix}^T$.

**Lemma 9.** *For* $j = 1, 2, \ldots, k-1$

$$|\lambda I - X_j| = F_j.$$

**Proof.** Similar to the proof of Lemma 4. □

**Theorem 4**

(a) $\sigma\left(Q\left(\mathcal{G}\right)\right) = \left(\cup_{j \in \Omega - \Delta} \sigma\left(W_j\right)\right) \cup \sigma\left(W_k\right) \cup \left(\cup_{j \in \Delta} X_j\right)$.

(b) *For each* $j \in \Omega - \Delta$, *the multiplicity of the eigenvalues of* $W_j$ *and, for each* $j \in \Delta$, *the multiplicity of the eigenvalues of* $X_j$, *as eigenvalues of* $Q(\mathcal{G})$, *is at least* $(n_j - n_{j+1})$. *The multiplicity of the eigenvalues of* $W_k$ *is at least* $1$.

**Corollary 2**

(a) *The smallest signless Laplacian eigenvalue of* $\mathcal{G}$ *is*

$$\lambda_1\left(Q\left(\mathcal{G}\right)\right) = \min\left\{\lambda_1\left(W_k\right), \min_{j \in \Delta}\left\{\lambda_1\left(X_j\right)\right\}\right\}.$$

(b) *The spectral radius of* $Q(\mathcal{G})$ *is*

$$\rho\left(Q\left(\mathcal{G}\right)\right) = \rho\left(W_k\right).$$

**Proof**

(a) Clearly our graph $\mathcal{G}$ is connected and contains odd cycles. Then, from Lemma 1, part $(c)$, $\lambda_1(Q(\mathcal{G})) > 0$. Moreover, from Lemma 1, part $(b)$, $\lambda_1(W_k) < \lambda_1(W_j)$ for all $j$. Thus the result follows.

(b) We have $W_1 = w_1 + 2(m_1 - 1)u_1 = X_1 + m_1 u_1$ and, for $j = 2, 3, \ldots, k-1$, $W_j = X_j + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & m_j u_j \end{bmatrix}$. Then, from Lemma 1, part $(b)$ and part $(d)$, $\rho(X_j) < \rho(W_j) < s\rho(W_k)$ for $j = 1, 2, \ldots, k-1$. Thus the result follows. □

**Example 3.** For the graph $\mathcal{G}$ in Example 1, $k = 4$, $\Delta = \{1, 2\}$, $d_1 = 1$, $d_2 = 4$, $d_3 = 3$, $d_4 = 2$, $n_1 = 12$, $n_2 = 4$, $n_3 = 2$, $n_4 = 1$. In addition, $\Omega = \{1, 2, 3\}$. Then

$$\sigma\left(Q\left(\mathcal{G}\right)\right) = \sigma\left(W_3\right) \cup \sigma\left(W_4\right) \cup \sigma\left(X_1\right) \cup \sigma\left(X_2\right),$$

where

$$W_3 = \begin{bmatrix} w_1 + 4u_1 & \sqrt{3}w_1 & \\ \sqrt{3}w_1 & 3w_1 + w_2 + 2u_2 & \sqrt{2}w_2 \\ & \sqrt{2}w_2 & 2w_2 + w_3 \end{bmatrix},$$

$$W_4 = \begin{bmatrix} w_1 + 4u_1 & \sqrt{3}w_1 & & \\ \sqrt{3}w_1 & 3w_1 + w_2 + 2u_2 & \sqrt{2}w_2 & \\ & \sqrt{2}w_2 & 2w_2 + w_3 & \sqrt{2}w_3 \\ & & \sqrt{2}w_3 & 2w_3 \end{bmatrix},$$

$$X_1 = [w_1 + u_1], \quad X_2 = \begin{bmatrix} w_1 + 4u_1 & \sqrt{3}w_1 \\ \sqrt{3}w_1 & 3w_1 + w_2 \end{bmatrix}.$$

For $w_1 = 2,\ w_2 = 2.5,\ w_3 = 3,\ u_1 = 1.5$ and $u_2 = 2$, the eigenvalues of $Q(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $W_3$ : | 4.8129 | 8 | 15.6871 | | $n_3 - n_4 = 1$ |
| $W_4$ : | 2.0125 | 6.1031 | 10.3742 | 16.0103 | 1 |
| $X_1$ : | 3.5 | | | | $n_1 - n_2 = 8$ |
| $X_2$ : | 4.7769 | 11.7231 | | | $n_2 - n_3 = 2$ |

We now search for the spectrum of the adjacency matrix of $\mathcal{G}$.

**Definition 7.** Let

$$G_0(\lambda) = 1, \quad G_1(\lambda) = \lambda - (m_1 - 1)\, u_1,$$
$$G_j(\lambda) = \left(\lambda - (m_j - 1)\, u_j\right) G_{j-1}(\lambda) - m_{j-1} w_{j-1}^2 G_{j-2}(\lambda)$$

for $j = 2, 3, \ldots, k-1$, and

$$G_k(\lambda) = \lambda G_{k-1}(\lambda) - m_{k-1} w_{k-1}^2 G_{k-2}(\lambda).$$

**Theorem 5.** *The characteristic polynomial of $A(\mathcal{G})$ is*

$$|\lambda I - A(\mathcal{G})| = G_k(\lambda) \prod_{j \in \Omega} \left(G_j(\lambda) + m_j u_j G_{j-1}(\lambda)\right)^{n_j - n_{j+1}}.$$

*Then*

$$|\lambda I - A(\mathcal{G})| = G_k(\lambda) \prod_{j \in \Omega - \Delta} G_j^{n_j - n_{j+1}}(\lambda) \prod_{j \in \Delta} \left(G_j(\lambda) + m_j u_j G_{j-1}(\lambda)\right)^{n_j - n_{j+1}}. \tag{10}$$

**Proof.** Applying Lemma 7 to the matrix $\lambda I - A(\mathcal{G})$, we obtain

$$\beta_1 - (m_1 - 1)\, u_1 = \frac{G_1}{G_0} \neq 0$$

$$\beta_1 + u_1 = \frac{G_1}{G_0} + m_1 u_1,$$

$$\beta_j - (m_j - 1)\, u_j = \frac{G_j}{G_{j-1}} \neq 0$$

$$\beta_j + u_j = \frac{G_j}{G_{j-1}} + m_j u_j,$$

for $j = 2, 3, \ldots, k-1$, and

$$\beta_k = \frac{G_k}{G_{k-1}}.$$

Then, replacing these equalities in (8), the theorem follows. $\square$

**Definition 8.** For $j = 1, 2, 3, \ldots, k-1$, let $Y_j$ be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$Y_k = \begin{bmatrix} (m_1 - 1)\, u_1 & \sqrt{m_1} w_1 & & & \\ \sqrt{m_1} w_1 & (m_2 - 1)\, u_2 & \ddots & & \\ & \ddots & \ddots & \sqrt{m_{k-2}} w_{k-2} & \\ & & \sqrt{m_{k-2}} w_{k-2} & (m_{k-1} - 1)\, u_{k-1} & \sqrt{m_{k-1}} w_{k-1} \\ & & & \sqrt{m_{k-1}} w_{k-1} & 0 \end{bmatrix}.$$

**Lemma 10.** *For* $j = 1, 2, \ldots, k$,

$$\det\left(\lambda I - Y_j\right) = G_j.$$

**Proof.** Similar to the proof of Lemma 4. $\square$

The polynomials

$$H_j = G_j + m_j u_j G_{j-1}$$

appear in (10). We have

$$G_j(\lambda) = \left(\lambda - (m_j - 1)\, u_j\right) G_{j-1}(\lambda) - m_{j-1} w_{j-1}^2 G_{j-2}(\lambda),$$
$$H_1 = \lambda - (m_1 - 1)\, u_1 + m_1 u_1 = \lambda + u_1$$

and

$$\begin{aligned} H_j &= \left(\lambda - (m_j - 1)\, u_j\right) G_{j-1} - m_{j-1} w_{j-1}^2 G_{j-2} + m_j u_j G_{j-1} \\ &= \left(\lambda + u_j\right) G_{j-1} - m_{j-1} w_{j-1}^2 G_{j-2} \end{aligned}$$

for $j = 2, 3, \ldots, k-1$.

**Definition 9.** Let

$$\begin{aligned} Z_1 &= [-u_1] \\ Z_j &= \begin{bmatrix} Y_{j-1} & \mathbf{v}_{j-1} \\ \mathbf{v}_{j-1}^T & -u_j \end{bmatrix} \\ &= Y_j - \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & m_j u_j \end{bmatrix} \quad (2 \leqslant j \leqslant k-1). \end{aligned}$$

where $\mathbf{v}_{j-1}$ is the $(j-1)$-dimensional vector $\mathbf{v}_{j-1} = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{m_{j-1}} w_{j-1} \end{bmatrix}^T$.

**Lemma 11.** *For* $j = 1, 2, \ldots, k-1$

$$\left|\lambda I - Z_j\right| = H_j.$$

**Proof.** Similar to the proof of Lemma 4. $\square$

**Theorem 6**

(a) $\sigma\left(A\left(\mathcal{G}\right)\right) = \sigma\left(Y_k\right) \cup \left(\cup_{j \in \Omega - \Delta} \sigma\left(Y_j\right)\right) \cup \left(\cup_{j \in \Delta} Z_j\right)$.

(b) *For each* $j \in \Omega - \Delta$, *the multiplicity of the eigenvalues of* $Y_j$ *and, for each* $j \in \Delta$, *the multiplicity of the eigenvalues of* $Z_j$, *as eigenvalues of* $A(\mathcal{G})$, *is at least* $(n_j - n_{j+1})$. *The multiplicity of the eigenvalues of* $Y_k$ *is at least* 1.

**Corollary 3**

$$\rho\left(A\left(\mathcal{G}\right)\right) = \rho\left(Y_k\right).$$

**Proof.** We have $Y_1 = (m_1 - 1)u_1 = Z_1 + m_1 u_1$ and, for $j = 2, 3, \ldots, k-1$, $Y_j = Z_j + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & m_j u_j \end{bmatrix}$. Then, from Lemma 1, part $(a)$ and part $(b)$, $\rho(Z_j) \leqslant \rho(Y_j) < \rho(Y_k)$ for $j = 1, 2, \ldots, k-1$. Thus the result follows. $\square$

**Example 4.** For the graph $\mathcal{G}$ in Example 1, $k = 4$, $\Delta = \{1, 2\}$, $d_1 = 1$, $d_2 = 4$, $d_3 = 3$, $d_4 = 2$, $n_1 = 12$, $n_2 = 4$, $n_3 = 2$, $n_4 = 1$. and $\Omega = \{1, 2, 3\}$. Then

$$\sigma\left(A\left(\mathcal{G}\right)\right) = \sigma\left(Y_3\right) \cup \sigma\left(Y_4\right) \cup \sigma\left(Z_1\right) \cup \sigma\left(Z_2\right)$$

where

$$Y_3 = \begin{bmatrix} 2u_1 & \sqrt{3}w_1 & \\ \sqrt{3}w_1 & u_2 & \sqrt{2}w_2 \\ & \sqrt{2}w_2 & 0 \end{bmatrix}$$

$$Y_4 = \begin{bmatrix} 2u_1 & \sqrt{3}w_1 & & \\ \sqrt{3}w_1 & u_2 & \sqrt{2}w_2 & \\ & \sqrt{2}w_2 & 0 & \sqrt{2}w_{k-1} \\ & & \sqrt{2}w_{k-1} & 0 \end{bmatrix}$$

$$Z_1 = [-u_1],\ Z_2 = \begin{bmatrix} 2u_1 & \sqrt{3}w_1 \\ \sqrt{3}w_1 & -u_2 \end{bmatrix}.$$

For $w_1 = 2$, $w_2 = 2.5$, $w_3 = 3$, $u_1 = 1.5$ and $u_2 = 2$, the eigenvalues of $A(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $Y_3$ : | $-3.4644$ | $1.5700$ | $6.8944$ | | $n_3 - n_4 = 1$ |
| $Y_4$ : | $-5.4132$ | $-0.7166$ | $3.7972$ | $7.3325$ | $1$ |
| $Z_1$ : | $-1.5$ | | | | $n_1 - n_2 = 8$ |
| $Z_2$ : | $-3.7720$ | $4.7720$ | | | $n_2 - n_3 = 2$ |

## 4. The unweighted case and some particular graphs

From Theorem 2, 4 and 6, we obtain

**Corollary 4.** *If* $w_1 = w_2 = \ldots = w_{k-1} = 1$ *and* $u_j = 1$ *for all* $j \in \Delta$ *then*

(a) $\sigma\left(L\left(\mathcal{G}\right)\right) = \sigma\left(U_k\right) \cup \left(\cup_{j \in \Omega - \Delta} \sigma\left(U_j\right)\right) \cup \left(\cup_{j \in \Delta} V_j\right)$,

*where* $U_j$ *is the* $j \times j$ *leading principal submatrix of the* $k \times k$ *matrix*

$$U_k = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & d_2 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{k-1} - 1} & \\ & & \sqrt{d_{k-1} - 1} & d_{k-1} & \sqrt{d_k} \\ & & & \sqrt{d_k} & d_k \end{bmatrix}$$

*and*

$$V_1 = [d_2], \quad V_j = \begin{bmatrix} U_{j-1} & \mathbf{v}_{j-1} \\ \mathbf{v}_{j-1}^T & d_j + d_{j+1} - 1 \end{bmatrix} \quad (2 \leqslant j \leqslant k-2),$$

$$V_{k-1} = \begin{bmatrix} U_{k-2} & \mathbf{v}_{k-2} \\ \mathbf{v}_{k-2}^T & d_{k-1} + d_k \end{bmatrix}.$$

(b) $\sigma\left(Q\left(\mathcal{G}\right)\right) = \sigma\left(W_k\right) \cup \left(\cup_{j \in \Omega - \Delta} \sigma\left(W_j\right)\right) \cup \left(\cup_{j \in \Delta} X_j\right)$,
*where $W_j$ is the $j \times j$ leading principal submatrix of the $k \times k$ matrix*

$$W_k = \begin{bmatrix} 2d_2 - 3 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & d_2 + 2\left(d_3 - 2\right) & & & \\ & & \ddots & & \\ & & d_{k-2} + 2\left(d_{k-1} - 2\right) & \sqrt{d_{k-1} - 1} & \\ & & \sqrt{d_{k-1} - 1} & d_{k-1} + 2\left(d_k - 1\right) & \sqrt{d_k} \\ & & & \sqrt{d_k} & d_k \end{bmatrix}$$

*and*

$$X_1 = d_2 - 2, \quad X_j = \begin{bmatrix} W_{j-1} & \mathbf{v}_{j-1} \\ \mathbf{v}^T & d_j + d_{j+1} - 3 \end{bmatrix} \quad (2 \leqslant j \leqslant k-2),$$

$$X_{k-1} = \begin{bmatrix} W_{k-2} & \mathbf{v}_{k-2} \\ \mathbf{v}_{k-2}^T & d_{k-1} + d_k - 2 \end{bmatrix}.$$

(c) $\sigma\left(A\left(\mathcal{G}\right)\right) = \sigma\left(Y_k\right) \cup \left(\cup_{j \in \Omega - \Delta} \sigma\left(Y_j\right)\right) \cup \left(\cup_{j \in \Delta} Z_j\right)$
*where $Y_j$ is the $j \times j$ leading principal submatrix of the $k \times k$ matrix*

$$Y_k = \begin{bmatrix} d_2 - 2 & \sqrt{d_2 - 1} & & & & \\ \sqrt{d_2 - 1} & d_3 - 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & d_{k-1} - 2 & \sqrt{d_{k-1} - 1} & \\ & & & \sqrt{d_{k-1} - 1} & d_k - 1 & \sqrt{d_k} \\ & & & & \sqrt{d_k} & 0 \end{bmatrix}$$

*and*

$$Z_1 = [-u_1], \quad Z_j = \begin{bmatrix} Y_{j-1} & \mathbf{v}_{j-1} \\ \mathbf{v}_{j-1}^T & -1 \end{bmatrix} \quad (2 \leqslant j \leqslant k-1).$$

*The multiplicities of the eigenvalues of $L(\mathcal{G}), Q(\mathcal{G})$ and $A(\mathcal{G})$ are given in Theorem 2, 4 and 6, respectively.*

**Example 5.** Suppose that $\mathcal{G}$ in Example 1 is an unweighted graph. We have $k = 4$, $\Delta = \{1, 2\}$, $d_1 = 1$, $d_2 = 4$, $d_3 = 3$, $d_4 = 2$, $n_1 = 12$, $n_2 = 4$, $n_3 = 2$, $n_4 = 1$ and $\Omega = \{1, 2, 3\}$.

From Corollary 4, part $(a)$

$$\sigma\left(L\left(\mathcal{G}\right)\right)=\sigma\left(U_3\right)\cup\sigma\left(U_4\right)\cup\sigma\left(V_1\right)\cup\sigma\left(V_2\right),$$

$$U_3=\begin{bmatrix}1 & \sqrt{3} & \\ \sqrt{3} & 4 & \sqrt{2}\\ & \sqrt{2} & 3\end{bmatrix},\quad U_4=\begin{bmatrix}1 & \sqrt{3} & & \\ \sqrt{3} & 4 & \sqrt{2} & \\ & \sqrt{2} & 3 & \sqrt{2}\\ & & \sqrt{2} & 2\end{bmatrix},$$

$$V_1=[4],\quad V_2=\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & 6\end{bmatrix}.$$

The eigenvalues of $L(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $U_3$ : | 0.0746 | 2.4481 | 5.4774 | | $n_3-n_4=1$ |
| $U_4$ : | 0 | 1 | 3.3820 | 5.6180 | 1 |
| $V_1$ : | 4 | | | | $n_1-n_2=8$ |
| $V_2$ : | 0.4586 | 6.5414 | | | $n_2-n_3=2$ |

From Corollary 4, part $(b)$

$$\sigma\left(Q\left(\mathcal{G}\right)\right)=\sigma\left(W_3\right)\cup\sigma\left(W_4\right)\cup\sigma\left(X_1\right)\cup\sigma\left(X_2\right)$$

$$W_3=\begin{bmatrix}5 & \sqrt{3} & \\ \sqrt{3} & 6 & \sqrt{2}\\ & \sqrt{2} & 3\end{bmatrix},\quad W_4=\begin{bmatrix}5 & \sqrt{3} & & \\ \sqrt{3} & 6 & \sqrt{2} & \\ & \sqrt{2} & 3 & \sqrt{2}\\ & & \sqrt{2} & 2\end{bmatrix}$$

$$X_1=[2],\quad X_2=\begin{bmatrix}5 & \sqrt{3}\\ \sqrt{3} & 4\end{bmatrix}.$$

The eigenvalues of $Q(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $W_3$ : | 2.2485 | 4.1589 | | | $n_3-n_4=1$ |
| $W_4$ : | 0.8343 | 3.1422 | 4.4073 | 7.6162 | 1 |
| $X_1$ : | 2 | | | | $n_1-n_2=8$ |
| $X_2$ : | 2.6972 | 6.3028 | | | $n_2-n_3=2$ |

From Corollary 4, part $(c)$

$$\sigma\left(A\left(\mathcal{G}\right)\right)=\sigma\left(Y_3\right)\cup\sigma\left(Y_4\right)\cup\sigma\left(Z_1\right)\cup\sigma\left(Z_2\right)$$

$$Y_3=\begin{bmatrix}2 & \sqrt{3} & \\ \sqrt{3} & 1 & \sqrt{2}\\ & \sqrt{2} & 0\end{bmatrix},\quad Y_4=\begin{bmatrix}2 & \sqrt{3} & & \\ \sqrt{3} & 1 & \sqrt{2} & \\ & \sqrt{2} & 0 & \sqrt{2}\\ & & \sqrt{2} & 0\end{bmatrix}$$

$$Z_1=[-1],\quad Z_2=\begin{bmatrix}2 & \sqrt{3}\\ \sqrt{3} & -1\end{bmatrix}.$$

The eigenvalues of $A(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $Y_3$ : | −1.3615 | 0.8326 | 3.5289 | | $n_3-n_4=1$ |
| $Y_4$ : | −1.9444 | −0.1849 | 1.5572 | 3.5722 | 1 |
| $Z_1$ : | −1 | | | | $n_1-n_2=8$ |
| $Z_2$ : | −1.7913 | 2.7913 | | | $n_2-n_3=2$ |

Finally we study some particular cases. From Theorem 2, 4 and 6, we have

**Corollary 5**

(a) *If* $\Delta = \Omega$ *then*

$$\sigma\left(L\left(\mathcal{G}\right)\right) = \sigma\left(U_k\right) \cup \left(\cup_{j\in\Omega} V_j\right),$$
$$\sigma\left(Q\left(\mathcal{G}\right)\right) = \sigma\left(W_k\right) \cup \left(\cup_{j\in\Omega} X_j\right),$$
$$\sigma\left(A\left(\mathcal{G}\right)\right) = \sigma\left(Y_k\right) \cup \left(\cup_{j\in\Omega} Z_j\right).$$

(b) *If* $\Delta = \{1\}$ *then*

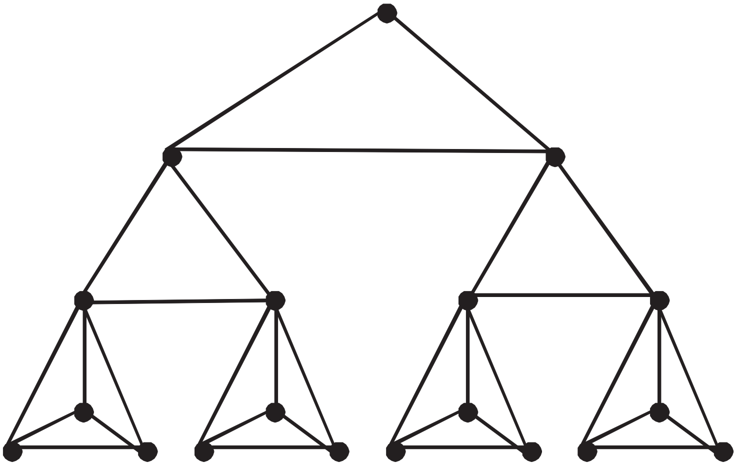
$$\sigma\left(L\left(\mathcal{G}\right)\right) = \sigma\left(U_k\right) \cup \left(\cup_{j\in\Omega-\{1\}}\sigma\left(U_j\right)\right) \cup V_1,$$
$$\sigma\left(Q\left(\mathcal{G}\right)\right) = \sigma\left(W_k\right) \cup \left(\cup_{j\in\Omega-\{1\}}\sigma\left(W_j\right)\right) \cup X_1,$$
$$\sigma\left(A\left(\mathcal{G}\right)\right) = \sigma\left(Y_k\right) \cup \left(\cup_{j\in\Omega-\{1\}}\sigma\left(Y_j\right)\right) \cup Z_1.$$

(c) *If* $\Delta = \{k-1\}$ *then*

$$\sigma\left(L\left(\mathcal{G}\right)\right) = \sigma\left(U_k\right) \cup \left(\cup_{j\in\Omega-\{k-1\}}\sigma\left(U_j\right)\right) \cup V_{k-1},$$
$$\sigma\left(Q\left(\mathcal{G}\right)\right) = \sigma\left(W_k\right) \cup \left(\cup_{j\in\Omega-\{k-1\}}\sigma\left(W_j\right)\right) \cup X_{k-1},$$
$$\sigma\left(A\left(\mathcal{G}\right)\right) = \sigma\left(Y_k\right) \cup \left(\cup_{j\in\Omega-\{k-1\}}\sigma\left(Y_j\right)\right) \cup Z_{k-1}.$$

Finally, we illustrate Corollary 5, part $(a)$, with an example.

**Example 6.** Consider the unweighted graph $\mathcal{G}$



For this graph, $\Delta = \{1, 2, 3\}$. From Corollary 5

$$\sigma\left(L\left(\mathcal{G}\right)\right) = \sigma\left(U_4\right) \cup \sigma\left(V_1\right) \cup \sigma\left(V_2\right) \cup \sigma\left(V_3\right),$$

$$U_4 = \begin{bmatrix} 1 & \sqrt{3} & & \\ \sqrt{3} & 4 & \sqrt{2} & \\ & \sqrt{2} & 3 & \sqrt{2} \\ & & \sqrt{2} & 2 \end{bmatrix},$$

$$V_1 = [4], \quad V_2 = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 6 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & \sqrt{3} & \\ \sqrt{3} & 4 & \sqrt{2} \\ & \sqrt{2} & 5 \end{bmatrix},$$

$$\sigma\left(Q\left(\mathcal{G}\right)\right)=\sigma\left(W_4\right)\cup\sigma\left(X_1\right)\cup\sigma\left(X_2\right)\cup\sigma\left(X_3\right),$$

$$W_4=\begin{bmatrix}5 & \sqrt{3} & & \\ \sqrt{3} & 6 & \sqrt{2} & \\ & \sqrt{2} & 5 & \sqrt{2} \\ & & \sqrt{2} & 2\end{bmatrix}$$

$$X_1=[2],\quad X_2=\begin{bmatrix}5 & \sqrt{3} \\ \sqrt{3} & 4\end{bmatrix},\quad V_3=\begin{bmatrix}5 & \sqrt{3} & \\ \sqrt{3} & 6 & \sqrt{2} \\ & \sqrt{2} & 3\end{bmatrix},$$

$$\sigma\left(A\left(\mathcal{G}\right)\right)=\sigma\left(Y_4\right)\cup\sigma\left(Z_1\right)\cup\sigma\left(Z_2\right)\cup\sigma\left(Z_3\right),$$

$$Y_4=\begin{bmatrix}2 & \sqrt{3} & & \\ \sqrt{3} & 1 & \sqrt{2} & \\ & \sqrt{2} & 1 & \sqrt{2} \\ & & \sqrt{2} & 0\end{bmatrix},$$

$$Z_1=[-1],\quad Z_2=\begin{bmatrix}2 & \sqrt{3} \\ \sqrt{3} & -1\end{bmatrix},\quad V_3=\begin{bmatrix}2 & \sqrt{3} & \\ \sqrt{3} & 1 & \sqrt{2} \\ & \sqrt{2} & -1\end{bmatrix}.$$

For instance, the eigenvalues of $L(\mathcal{G})$ are

| | | | | | *multiplicity* |
|---|---|---|---|---|---|
| $U_4$ : | 0 | 1 | 3.3820 | 5.6180 | 1 |
| $V_1$ : | 4 | | | | $n_1 - n_2 = 8$ |
| $V_2$ : | 0.4586 | 6.5414 | | | $n_2 - n_3 = 2$ |
| $V_3$ : | 0.1322 | 3.6502 | 6.2176 | | $n_3 - n_4 = 1$ |


## Acknowledgement

The authors wish to thank the referee for the valuable comments which led to an improved version of the paper.


## References


[1] D.M. Cardoso, D. Cvetković, P. Rowlinson, S.K. Simić, A sharp lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph,Linear Algebra Appl. 429 (2008) 2770–2780.
[2] D. Cvetković, P. Rowlinson, S.K. Simić, Signless Laplacian of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[3] D. Cvetković, S.K. Simić, Towards a Spectral Theory of Graphs based on the signless Laplacian I, Publ. Inst. Math. (Beograd) 85 (99) (2009) 19–33.
[4] D. Cvetković, S.K. Simić, Towards a Spectral Theory of Graphs based on the signless Laplacian II, Linear Algebra Appl. 432 (2010) 2257–2272.
[5] D. Cvetković, S.K. Simić, Towards a Spectral Theory of Graphs based on the signless Laplacian III, Appl. Anal. Discrete Math. 4 (2010) 156–166.
[6] O.R. Diestel, Graph Theory, Electronic Editions 2005, Springer-Verlag, Hiedlberg, New York, 2005.
[7] R. Fernandes, H. Gomes, E.A. Martins, On the spectra of some graphs like weighted rooted trees, Linear Algebra Appl. 428 (2008) 2654–2674.
[8] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298–305.
[9] G.H. Golub, C.F. Van Loan, Matrix Computations, second ed., Johns Hopkins University Press, Baltimore, 1989.
[10] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[11] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[12] O. Rojo, M. Robbiano, On the spectra of some weighted rooted trees and applications, Linear Algebra Appl. 420 (2007) 310–328.
[13] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Inc., 1965.
[14] F. Zhang, Matrix Theory, Springer, 1999.